Slow steady rotation of axially symmetric bodies in a viscous fluid

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The Stokes flow problem is considered here for the case in which an axially symmetric body is uniformly rotating about its axis of symmetry. Analytic solutions are presented for the heretofore unsolved cases of a spindle, a torus, a lens, and various special configurations of a lens. Formulas are derived for the angular velocity of the flow field and for the couple experienced by the body in each case.

1. Introduction

The value of the couple experienced by axially symmetric bodies, rotating steadily in a viscous and incompressible fluid, is needed in designing and calibrating viscometers. Therefore, many attempts have been made to evaluate such a couple for various bodies of revolution. When inertial effects can be validly ignored, so that Stokes's linearized theory applies, the solutions have been found for some configurations. These configurations are a sphere (Lamb 1945), spheroids and a pair of spheres (Jeffery 1915). Jeffery has also given the solution for a circular disk as a limiting case of the solution for an oblate spheroid. The value of the angular velocity of the flow has also been calculated for each of the above-mentioned cases by the same authors.

The purpose of this paper is to discuss the flow when a spindle, a torus and a lens are rotating steadily in an incompressible viscous fluid. Various special configurations of a lens, e.g. a hemisphere and a spherical cap, are also considered and explicit calculation of the couple has been made in each of these cases. Following other workers in the field, we assume the motion to be slow enough to justify the neglect of the inertial terms in the Navier–Stokes equations.

The procedure for solving the present problem is similar to the one given by Payne & Pell (1960). They have discussed the Stokes flow problem of axially symmetric bodies when the flow at points distant from the body is uniform and parallel to the axis of symmetry. They solve their problem with the help of the generalized axially symmetric potential theory. Furthermore, they derive a relation between the drag on a body and the stream function of the flow. In the present analysis we find a corresponding relation between the couple experienced by a body and the angular velocity of the flow. Furthermore, it is found in the following analysis that there is a relation between such a couple on a body and the polarization potential of that body.

The known cases of spheroids and a pair of spheres can be solved in a relatively simple way by this method. Since the simplicity of this method in solving these cases has been demonstrated by Payne & Pell, we omit the solutions for these configurations in the present work. The other two known cases, namely, a sphere and a circular disk, are briefly mentioned in the following work since they form special cases of a lens and thus provide a check on our analysis.

2. Equations of motion

Let the z-axis be the axis of symmetry and the x- and y-axes be mutually orthogonal axes in a plane perpendicular to the z-axis. Let u_i $(i = 1, 2, 3)$ denote the components of the velocity vector, p the pressure, and μ the coefficient of viscosity. Then for rotation about the z-axis, the Stokes-flow equations in a region $\mathscr{D}$ exterior to a closed boundary B may be expressed as

$$\mu \Delta u_i = p_{,i} \quad (u_{i,i} = 0), \tag{1}$$

where we have used the summation convention. On B

$$u_1 = -\omega_0 y, \quad u_2 = \omega_0 x, \quad u_3 = 0, \tag{2}$$

where ω_0 is the uniform angular velocity of the body of revolution.

These equations can be satisfied if we choose

$$u_1 \equiv -\omega_0 \phi_2, \quad u_2 \equiv \omega_0 \phi_1, \quad u_3 \equiv 0, \quad p = \text{constant}, \tag{3}$$

where

$$\left.\begin{aligned} \Delta\phi_1 = 0, \quad \Delta\phi_2 = 0, \quad \text{in} \quad \mathscr{D}, \\ \phi_1 = x, \qquad \phi_2 = y, \quad \text{on} \quad B. \end{aligned}\right\} \tag{4}$$

By this choice of u_i, the equation of continuity may be automatically satisfied.

Clearly ϕ_1 and ϕ_2 are polarization potentials (Schiffer & Szegö 1949) and are of the form

$$\phi_1 = \frac{x}{r^2}\,\Omega(r, z), \quad \phi_2 = \frac{y}{r^2}\,\Omega(r, z), \tag{5}$$

where $r^2 = x^2 + y^2$ and Ω satisfies the equation

$$L_{-1}\Omega \equiv \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)\Omega = 0. \tag{6}$$

This equation could also have been derived in a different way (Kanwal 1955). If ω denotes the angular velocity of the flow, then we observe that $\omega/\omega_0 = \Omega/r^2$. Moreover, from (4) we find that

$$\Omega = r^2 \text{ on } B. \tag{7}$$

3. Couple on an axially symmetric body

Payne & Pell have given an elegant formula for the drag experienced by an axially symmetric body in terms of the stream function. We now derive a similar formula for the couple in terms of the angular velocity of the flow. The value of the couple can, however, be found in another way also. It turns out that the couple on a body in the present problem can be related to the polarization of that body. Furthermore, since there is a tie between the polarization and the virtual mass of a body (Payne 1956), we can get a relation between the couple on a body and the virtual mass of that body. We first set out to investigate these relations.

Now the couple N is given by the expression

$$N = \frac{\mu}{2\omega_0}\iiint_{\mathscr{D}} (u_{i,j}+u_{j,i})\,(u_{i,j}+u_{j,i})\,dV, \tag{8}$$

where we have adopted the summation convention. If we put

$$D(u_i) = \iiint_{\mathscr{D}} |\mathrm{grad}\, u_i|^2\, dV, \tag{9}$$

we get
$$N = \frac{\mu}{\omega_0}\left[D(u_1)+D(u_2)+\iiint_{\mathscr{D}} (u_{i,j}u_{j,i})\,dV\right],$$

$$= \mu\omega_0[D(\phi_1)+D(\phi_2)] + \frac{\mu}{\omega_0}\iint_B u_i[u_{j,i}n_j - u_{j,j}n_i]\,dS, \tag{10}$$

where n_i are the components of the unit normal to B. The integral over B involves only u_i and its tangential derivatives so that we may replace u_i throughout by its boundary values. Thus

$$N = \mu\omega_0\left[D(\phi_1)+D(\phi_2)-\iint_B (xn_x+yn_y)\,dS\right],$$

$$= 2\mu\omega_0[P+V], \tag{11}$$

where V is the volume of the body B and $P = D(\phi_1) = D(\phi_2)$ is the suitably normalized polarization. Now Schiffer & Szegö (1949) have shown that

$$P+V = 4\pi e_x = 4\pi e_y, \tag{12}$$

where e_x and e_y are the dipole coefficients. Thus

$$N = 8\pi\mu\omega_0 e_x. \tag{13}$$

In their paper, Schiffer & Szegö have tabulated e_x for spheroids, spindle, torus, two spheres, and lenses, giving several particular examples for the lens.

If the meridian section ($r > 0$) of $\mathscr{D}$ is simply connected, then it has been shown by Payne (1956) that the virtual mass M, due to a potential flow along the z-axis, is given by the relation

$$P+V = 2(M+V). \tag{14}$$

Thus from (11) and (14) we get the relation between the couple and the virtual mass as

$$N = 4\mu\omega_0(M+V). \tag{15}$$

In a previous paper (1952) Payne has calculated the value of the virtual mass for the cases of a spindle and a lens.

We finally derive the value of the couple in terms of the angular velocity ω of the flow. The relation (8) can be written as

$$N = C + 4\mu\omega_0 V, \tag{16}$$

where
$$C = \frac{\mu}{2\omega_0}\iiint_{\mathscr{D}} (u_{i,j}-u_{j,i})\,(u_{i,j}-u_{j,i})\,dV. \tag{17}$$

In view of the axial symmetry we may restrict attention to any single meridian half plane ($r > 0$). Now let B^* denote the curve of intersection of the boundary B of the body with this half plane and let R be the region in ($r > 0$) bounded by B^* and a large semicircle Γ whose radius we shall ultimately let tend to infinity.

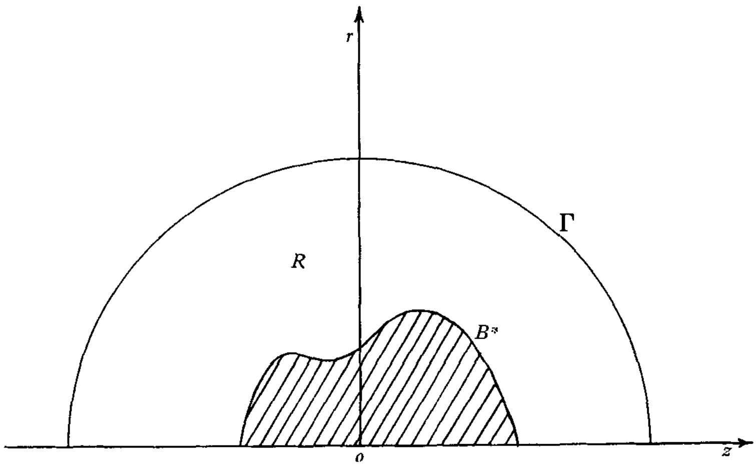

FIGURE 1. Configuration of flow field in meridian half plane.

The relation (17) then becomes

$$\frac{C}{2\pi\mu\omega_0} = \iint_R \frac{1}{r}\,|\operatorname{grad}\Omega|^2\,dr\,dz,$$

$$= \iint_R \frac{1}{r}\,\Omega L_{-1}\Omega\,dr\,dz + \oint_{B^*+\Gamma} \frac{1}{r}\,\Omega\frac{\partial\Omega}{\partial n}\,ds, \tag{18}$$

where the normal is directed outward from R. The integral along $r = 0$, between B^* and Γ vanishes since Ω is $O(r^2)$ as $r \to 0$ (Hyman 1954). Since $L_{-1}\Omega = 0$, the relation (18) gives

$$\frac{C}{2\pi\mu\omega_0} = \oint_{B^*+\Gamma} \frac{1}{r}\,r^2\frac{\partial\Omega}{\partial n}\,ds + \oint_{\Gamma} \frac{1}{r}\,(\Omega - r^2)\frac{\partial\Omega}{\partial n}\,ds. \tag{19}$$

But since both r^2 and Ω satisfy the equation (6), we have from Green's second identity

$$\oint_{B^*+\Gamma} \frac{1}{r}\,r^2\frac{\partial\Omega}{\partial n}\,ds = \oint_{B^*+\Gamma} \frac{1}{r}\,\Omega\frac{\partial r^2}{\partial n}\,ds. \tag{20}$$

From (19) and (20) we get

$$\frac{C}{2\pi\mu\omega_0} = 2\oint_{B^*} r^2\frac{\partial r}{\partial n}\,ds + \oint_{\Gamma} \frac{1}{r}\,(\Omega - r^2)\frac{\partial\Omega}{\partial n}\,ds + \oint_{\Gamma} \frac{1}{r}\,\Omega\frac{\partial r^2}{\partial n}\,ds. \tag{21}$$

Letting the radius of Γ tend to infinity and noting that as $\rho^2 = r^2 + z^2 \to \infty$

$$\Omega = \frac{br^2}{\rho^3} + O\left(\frac{1}{\rho^2}\right), \tag{22}$$

we have
$$\frac{C}{2\pi\mu\omega_0} = 2\oint_{B^*} r^2\frac{\partial r}{\partial n}ds + 3b\int_0^\pi \sin^3\theta\, d\theta,$$
$$= -4\iint_{\text{interior of } B^*} r\,dz\,dr + 4b,$$
$$= -\frac{2V}{\pi} + 4b. \tag{23}$$

Finally from (16), (22) and (23) we have our result:

$$N = 8\pi\mu\omega_0 \lim_{\rho\to\infty}\left[\rho^3\frac{\Omega}{r^2}\right], = 8\pi\mu \lim_{\rho\to\infty}[\rho^3\omega]. \tag{24}$$

It is of interest to compare it with the formula for the drag given by Payne & Pell (1960):

$$D = 8\pi\mu \lim_{\rho\to\infty}\left[\rho\frac{\psi^1}{r^2}\right].$$

We now turn to the consideration of a specific flow configuration.

4. The flow about a rotating spindle

A curve $\xi = \xi_0$ in dipolar co-ordinates defines the profile of a spindle. The dipolar transformation is given by

$$z + ir = ic\cot\tfrac{1}{2}(\xi + i\eta), \tag{25}$$

where c is a positive constant. The range of co-ordinates is chosen as $-\infty < \eta < \infty$, $0 < \xi \leqslant \pi$. The boundary of the spindle is given by $\xi = \xi_0 < \pi$, and the exterior region is defined by $0 < \xi < \xi_0$. Let us put

$$s = \cosh\eta, \quad t = \cos\xi.$$

We assume the solution of the equation (6) in the form

$$\Omega = r^2(s-t)^{-\frac{3}{2}}\int_0^\infty A(\alpha)\,K_\alpha^{(1)}(t)\cos\alpha\eta\,d\alpha, \tag{26}$$

where $^{(q)}$ denotes the qth partial derivative with respect to the argument and $K_\alpha(t)$ is a Legendre function of a complex degree commonly called a conal function (Hobson 1931, p. 445). It is defined as

$$K_\alpha(t) = \frac{2}{\pi}\cosh\alpha\pi\int_0^\infty (2\cosh u + 2\cos\xi)^{-\frac{1}{2}}\cos\alpha u\,du. \tag{27}$$

If we replace ξ by $(\pi-\xi)$ in (27), we obtain

$$K_\alpha(-t) = \frac{2}{\pi}\cosh\alpha\pi\int_0^\infty [2\cosh u - 2\cos\xi]^{-\frac{1}{2}}\cos\alpha u\,du. \tag{28}$$

The boundary condition gives the value of $A(\alpha)$ occurring in the equation (26), and the complete solution to our problem is

$$\Omega = 2^{\frac{3}{2}}(s-t)^{\frac{3}{2}}r^2\int_0^\infty \frac{K_\alpha^{(1)}(-t_0)\,K_\alpha^{(1)}(t)\cos\alpha\eta}{K_\alpha^{(1)}(t_0)\cosh\alpha\pi}d\alpha. \tag{29}$$

The couple experienced by the spindle is given by the formula

$$N = 8\pi\mu\omega_0 c^3\int_0^\infty \frac{(4\alpha^2+1)\,K_\alpha^{(1)}(-t_0)}{K_\alpha^{(1)}(t_0)\cosh\alpha\pi}\,d\alpha. \tag{30}$$

5. The flow about a rotating lens-shaped body

Let us introduce the peripolar transformation

$$z+ir = -c\cot\tfrac{1}{2}(\xi+i\eta), \tag{31}$$

where c is a positive constant. The profile of a lens is defined by two curves $\xi = \xi_1$, and $\xi = \xi_2$. We shall assume that $0 < \xi_1 < \xi_2 < 2\pi$. The external region is chosen as $\eta > 0$.

Let us assume the solution of the equation (6) in the form

$$\Omega = 2^{\frac{3}{2}}(s-t)^{\frac{3}{2}}\,r^2\int_0^\infty F(\alpha,\xi)\,\mathrm{sech}\,\alpha\pi K_\alpha^{(1)}(s)\,d\alpha, \tag{32}$$

where the quantities s, t and $^{(a)}$ are the same as defined in the previous section. The function $K_\alpha(s)$ is

$$K_\alpha(s) = \frac{2}{\pi}\cosh\alpha\pi\int_0^\infty [2\cosh u+2\cosh\eta]^{-\frac{1}{2}}\cos\alpha u\,du. \tag{33}$$

Furthermore, the quantity $F(\alpha,\xi)$ is assumed to be of the form

$$F(\alpha,\xi) = \{A(\alpha)\cosh\alpha\xi+B(\alpha)\sinh\alpha\xi\}. \tag{34}$$

It can be shown that (Hobson 1931, p. 451)

$$(s-t)^{-\frac{1}{2}} = \sqrt{2}\int_0^\infty \frac{\cosh\alpha(\xi-\pi)}{\cosh\alpha\pi}K_\alpha(s)\,d\alpha. \tag{35}$$

If we differentiate this relation with respect to s, we obtain

$$[2(s-t)]^{-\frac{3}{2}} = -\int_0^\infty \frac{\cosh\alpha(\xi-\pi)}{\cosh\alpha\pi}K_\alpha^{(1)}(s)\,d\alpha. \tag{36}$$

With the help of the above relation, the boundary condition provides the expression for $F(\alpha,\xi)$, as

$$\begin{aligned}&\sinh\alpha(2\pi-\xi_2+\xi_1)\,F(\alpha,\xi)\\ &\quad= \sinh\alpha(\xi_1-\xi)\cosh\alpha(\pi-\xi_2)+\cosh\alpha(\pi-\xi_1)\sinh\alpha(2\pi-\xi_2+\xi).\end{aligned} \tag{37}$$

The corresponding couple is

$$N = 8\pi\mu\omega_0c^3\int_0^\infty \frac{\{\sinh\alpha\xi_1\cosh\alpha(\pi-\xi_2)+\cosh\alpha(\pi-\xi_1)\sinh\alpha(2\pi-\xi_2)\}(4\alpha^2+1)}{\sinh\alpha(2\pi+\xi_1-\xi_2)\cosh\alpha\pi}\,d\alpha. \tag{38}$$

6. Special cases of the lens

(*a*) *Hemisphere*. In this case $\xi_1 = \frac{1}{2}\pi$, $\xi_2 = \pi$. The expression (37) becomes

$$\sinh\left(\frac{3\alpha\pi}{2}\right)F(\alpha,\xi) = \sinh\alpha\left(\frac{\pi}{2}-\xi\right)+\cosh\frac{\alpha\pi}{2}\sinh\alpha(\pi+\xi). \tag{39}$$

Similarly, the expression for the couple is obtained as

$$\begin{aligned}N &= \frac{8(135-59\sqrt{3})}{81}\pi\mu\omega_0c^3,\\ &= 10{\cdot}18\mu\omega_0c^3.\end{aligned} \tag{40}$$

(*b*) *The symmetrical* (*biconvex*) *lens*. In this instance, if $\xi = \xi_1$ is one face of the lens, then the other is $\xi = \xi_2 = 2\pi - \xi_1$. Thus, from the expressions (37) and (38) we have the corresponding values of the quantity $F(\alpha, \xi)$, and the couple N, as

$$\cosh \alpha\xi_1 F(\alpha, \xi) = \cosh \alpha(\pi - \xi_1) \cosh \alpha\xi, \tag{41}$$

and
$$N = 8\pi\mu\omega_0 c^3 \int_0^\infty (4\alpha^2 + 1)(1 - \tanh \alpha\pi \tanh \alpha\xi_1)\, d\alpha. \tag{42}$$

In the case of a sphere $\xi_1 = 2\pi$, so the expression (41) gives

$$F(\alpha, \xi) = \cosh \alpha\xi.$$

Thus from the equation (32), we obtain

$$\Omega = 2^{\frac{3}{2}}(s-t)^{\frac{3}{2}} r^2 \int_0^\infty \cosh \alpha\xi \operatorname{sech} \alpha\pi K_\alpha^{(1)}(s)\, d\alpha. \tag{43}$$

If we put ξ for $(\xi - \pi)$ in the relation (36), we readily derive the result

$$[2(s+t)]^{-\frac{3}{2}} = -\int_0^\infty \cosh \alpha\xi \operatorname{sech} \alpha\pi K_\alpha^{(1)}(s)\, d\alpha. \tag{44}$$

From (43) and (44), we finally have

$$\Omega = -r^2 \frac{(s-t)^{\frac{3}{2}}}{(s+t)^{\frac{3}{2}}}. \tag{45}$$

If a is the radius of the sphere, then $(\rho/a)^2 = (s+t)/(s-t)$. Therefore, we get the well-known results

$$\Omega = r^2 \frac{a^3}{\rho^3}, \tag{46}$$

and
$$N = 8\pi\mu\omega_0 a^3. \tag{47}$$

(*c*) *Spherical cap*. If $\xi_2 = \xi_1$, the two bounding surfaces of the lens coincide, the body becomes a portion of a spherical surface bounded by a circle of latitude and we have a spherical cap. The expression for $F(\alpha, \xi)$ is given by the equation (37) with $\xi_2 = \xi_1$. The value of the couple is

$$N = 4\pi\mu\omega_0 c^3 \left\{ f''(\xi_1) + f(\xi_1) + \frac{1}{3\pi} \right\}, \tag{48}$$

where
$$f(\xi_1) = \frac{1}{\pi} + \left(1 - \frac{\xi_1}{\pi}\right) \frac{1}{\sin \xi_1}. \tag{49}$$

When $\xi_1 \to \pi$, we get the case of the circular disk and N becomes

$$N = \tfrac{32}{3}\pi\mu\omega_0 c^3, \tag{50}$$

which is well known. Finally, when $\xi_1 \to \frac{1}{2}\pi$, the cap becomes hemispherical and the couple in this case is given as

$$N = 8\pi\mu\omega_0 c^3 \left(\frac{1}{2} + \frac{2}{3\pi}\right). \tag{51}$$

7. The flow about a rotating torus

In order to calculate the flow about a torus, we introduce toroidal co-ordinates ξ, η in a meridian plane by the transformation

$$\left.\begin{aligned} x &= \frac{c \sin \xi}{s-t}, \quad r = \frac{c \sinh \eta}{s-t}, \\ s &= \cosh \eta, \quad t = \cos \xi. \end{aligned}\right\} \tag{52}$$

The curves η = constant in $r \geqslant 0$ are circles which nest about $(0, c)$. Hence, any curve $\eta = \eta_0$ = constant, defines the boundary of a torus whose exterior is given by

$$\eta_0 \geqslant \eta \geqslant 0, \quad 0 \leqslant \xi < 2\pi.$$

We assume the solution to be given as

$$\Omega = r^2(s-t)^{\frac{3}{2}} \sum_{n=0}^{\infty}{}' A_n P^{(1)}_{n-\frac{1}{2}}(s) \cos n\xi, \tag{53}$$

where Σ' indicates that the term for $n = 0$ is to be multiplied by the factor $\frac{1}{2}$. The boundary condition gives

$$(s_0-t)^{-\frac{3}{2}} = \sum_{n=0}^{\infty}{}' A_n P^{(1)}_{n-\frac{1}{2}}(s_0) \cos n\xi. \tag{54}$$

If we differentiate the relation (Hobson 1931, p. 443)

$$(s-t)^{-\frac{1}{2}} = \frac{2\sqrt{2}}{\pi} \sum_{n=0}^{\infty}{}' Q_{n-\frac{1}{2}}(s) \cos n\xi, \tag{55}$$

with respect to s, we have

$$(s-t)^{-\frac{3}{2}} = -\frac{4\sqrt{2}}{\pi} \sum_{n=0}^{\infty}{}' Q^{(1)}_{n-\frac{1}{2}}(s) \cos n\xi. \tag{56}$$

From (54) and (56) we obtain the value of A_n as

$$A_n = -\frac{4\sqrt{2}}{\pi} \frac{Q^{(1)}_{n-\frac{1}{2}}(s_0)}{P^{(1)}_{n-\frac{1}{2}}(s_0)}. \tag{57}$$

The couple experienced by the torus is

$$N = -16\mu\omega_0 c^3 \sum_{n=0}^{\infty}{}' (4n^2-1) \frac{Q^{(1)}_{n-\frac{1}{2}}(s_0)}{P^{(1)}_{n-\frac{1}{2}}(s_0)}. \tag{58}$$

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